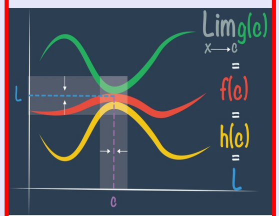
Calculus I

Lecture 14



Feb 19-8:47 AM

Class Quiz 6

Use ε and δ definition of a limit

to prove $\lim_{x \to 2} (x^2+2x-7)=1$, then for $\varepsilon=1$, what is your δ?

$f(x)=x^2+2x-7$, $a=2$, $L=1$ ✓

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|x^2+2x-7-1|<\varepsilon$ 〃 $|x-2|<\delta$

$|x^2+2x-8|<\varepsilon$ 〃 $|x-2|<\delta$

$|(x+4)(x-2)|<\varepsilon$

$\underbrace{|x+4|}_{\text{Bound}}\,\underbrace{|x-2|}_{\text{Keep}}<\varepsilon$

If $\delta \le 1$, $|x-2|<1$

$-1<x-2<1$

Add 6

$-1+6<x-2+6<1+6$

$5<x+4<7 \to |x+4|<7$

$|x+4||x-2|<7|x-2|<\varepsilon \to |x-2|<\frac{\varepsilon}{7}$

Choose $\delta=\min\left\{1,\frac{\varepsilon}{7}\right\}$

If $\varepsilon=1$
$\delta=\min\left\{1,\frac{1}{7}\right\}=\boxed{\frac{1}{7}}$

Sep 18-7:00 AM

For $\varepsilon = 1$, find a $\delta > 0$ such that

$\lim_{x \to 1} (x^3 + 5x) = 6.$

$f(x) = x^3 + 5x$ $\quad a = 1$ $\quad L = 6$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^3 + 5x - 6| < \varepsilon$ $\quad$ $|x - 1| < \delta$

$|(x^2 + x + 6)(x - 1)| < \varepsilon$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 5 & -6 \\ & & 1 & 1 & 6 \\ \hline & 1 & 1 & 6 & 0 \end{array}$$

$\underbrace{|x^2 + x + 6|}_{\text{Bound}}\ \underbrace{|x - 1|}_{\text{Keep}} < \varepsilon$

If $\delta \le 1$ then $|x - 1| < 1$

$-1 < x - 1 < 1$

$0 < x < 2$

If $x = 0$, $x^2 + x + 6 = 6$

If $x = 2$, $x^2 + x + 6 = 12$ $\rightarrow$ $6 < x^2 + x + 6 < 12$

$|x^2 + x + 6| < 12$

$|x^2 + x + 6|\,|x - 1| < 12|x - 1| < \varepsilon$

$|x - 1| < \frac{\varepsilon}{12}$

$\delta = \min\left\{1, \frac{\varepsilon}{12}\right\}$

when $\varepsilon = 1$ $\rightarrow$ $\delta = \min\left\{1, \frac{1}{12}\right\} = \boxed{\frac{1}{12}}$

Sep 18-7:47 AM

For $a > 0$, Prove $\lim_{x \to a} \sqrt{x} = \sqrt{a}$

$f(x) = \sqrt{x}$ $\quad a = a$ $\quad L = \sqrt{a}$ ✓

$[0, \infty)$

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|\sqrt{x} - \sqrt{a}| < \varepsilon$ $\quad$ $|x - a| < \delta$

$\left| \frac{\sqrt{x} - \sqrt{a}}{1} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right| < \varepsilon$

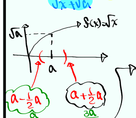
$\left| \frac{x - a}{\sqrt{x} + \sqrt{a}} \right| < \varepsilon$

$\frac{|x - a|}{|\sqrt{x} + \sqrt{a}|} < \varepsilon$

$\underbrace{\frac{1}{\sqrt{x} + \sqrt{a}}}_{\text{Bound}}\ \underbrace{|x - a|}_{\text{Keep}} < \varepsilon$

If $\frac{1}{\sqrt{x} + \sqrt{a}} < C$, $\frac{1}{\sqrt{x} + \sqrt{a}}|x - a| < C|x - a| < \varepsilon$

$|x - a| < \frac{\varepsilon}{C}$



If $\delta \le \frac{1}{2}a$ $\rightarrow$ $|x - a| < \frac{a}{2}$

$-\frac{a}{2} < x - a < \frac{a}{2}$

Add a

$-\frac{a}{2} + a < x < \frac{a}{2} + a$

$\frac{a}{2} < x < \frac{3a}{2}$

$\sqrt{\frac{a}{2}} < \sqrt{x} < \sqrt{\frac{3a}{2}}$

$\sqrt{\frac{a}{2}} + \sqrt{a} < \sqrt{x} + \sqrt{a} < \sqrt{\frac{3a}{2}} + \sqrt{a}$

$\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{a}{2}} + \sqrt{a}\right)\varepsilon\right\}$

$\frac{1}{\sqrt{\frac{a}{2}} + \sqrt{a}} > \frac{1}{\sqrt{x} + \sqrt{a}} > \frac{1}{\sqrt{\frac{3a}{2}} + \sqrt{a}}$

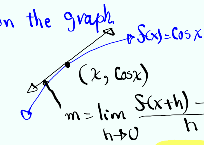
$\frac{1}{\sqrt{\frac{3a}{2}} + \sqrt{a}} < \frac{1}{\sqrt{x} + \sqrt{a}} < \frac{1}{\sqrt{\frac{a}{2}} + \sqrt{a}} \leftarrow C$

$\frac{\varepsilon}{C} = \frac{\varepsilon}{\frac{1}{\sqrt{\frac{a}{2}} + \sqrt{a}}}$

$= \left(\sqrt{\frac{a}{2}} + \sqrt{a}\right)\varepsilon$

Sep 18-7:59 AM

Find a formula for the slope of tangent line to the graph of $f(x)=\cos x$ at any point on the graph.



Recall
$\cos(A+B)=$
$\cos A\cos B-\sin A\sin B$

$$m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=\lim_{h\to 0}\left[\frac{\cos x(\cos h-1)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=\lim_{h\to 0}\frac{\cos x(\cos h-1)}{h}-\lim_{h\to 0}\frac{\sin x\sin h}{h}$$

$$=\cos x\cdot\boxed{\lim_{h\to 0}\frac{\cos h-1}{h}}_{\to 0}-\sin x\cdot\boxed{\lim_{h\to 0}\frac{\sin h}{h}}_{\to 1}$$

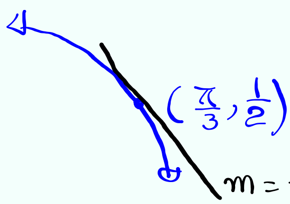
$$=\cos x\cdot 0-\sin x\cdot 1=\boxed{-\sin x}$$

If $f(x)=\sin x$, then $m_{\text{tan. line at any pt}}=\cos x$

If $f(x)=\cos x$, then $m_{\text{tan. line at any pt}}=-\sin x$

Sep 18-8:19 AM

Find equation of tan. line to the graph of $f(x)=\cos x$ at $x=\frac{\pi}{3}$.



$$m=-\sin x\Big|_{x=\pi/3}=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$$

$$y-y_1=m(x-x_1)$$

$$y-\frac{1}{2}=-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)$$

$$\boxed{y=-\frac{\sqrt{3}}{2}x+\frac{\sqrt{3}\pi}{6}+\frac{1}{2}}$$

Sep 18-8:30 AM